Existence of $(\Phi \otimes \Psi)$ bounded solutions of linear first-order Kronecker product systems of differential equations

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Abstract

This paper presents a criterion for the existence of $(\Phi \otimes \Psi)$ bounded solutions of the non-homogeneous first order Kronecker product system. This study enables to discuss stability and asymptotic stability of two linear systems of first order in a single framework.

Keywords: Φ and Ψ bounded solutions, Kronecker product of matrices, fundamental matrix, Lebesgue integrable functions

1. Introduction

In this paper, we shall be concerned with the $(\Phi \otimes \Psi)$ bounded solutions of the system of two first order linear systems of differential equations of different orders given by

$$x' = A(t)x \tag{1.1}$$

and

$$y' = B(t)y \tag{1.2}$$

where A is an $(n \times n)$ matrix and B is an $(m \times m)$ matrix and x, y are column vectors of orders $(n \times 1)$, $(m \times 1)$ respectively. Kronecker product of linear systems and its applications in two-point boundary value problems was first introduced by Murty and Fausett [14] in 2002. Many results followed after this basic paper in control theory and in systems analysis in [7]. In this paper, we present a criterion for the existence of $(\Phi \otimes \Psi)$ bounded solutions of the Kronecker product system defined by

$$\begin{aligned}(x \otimes y)' &= (x' \otimes y) + (x \otimes y') \\ &= (Ax \otimes y) + (x \otimes By) \\ &= [(A \otimes I_m) + (I_n \otimes B)][x \otimes y]\end{aligned} \tag{1.3}$$

where I is an identity matrix. Before we present our main results, we give the definition of Kronecker product of two matrices and their properties.

Definition 1.1 The Kronecker product of two matrices $A \in R^{m \times n}$ and $B \in R^{p \times q}$ is defined as

$$(A \otimes B) = (a_{ij} B)$$

for all $i = 1, 2, ..., m$ and $j = 1, 2, ..., n$ and is of order $R^{mp \times nq}$.

The Kronecker product of matrices defined above has the following properties:

(I) $(A\otimes B)(C\otimes D)=(AC\otimes BD)$ (provided AC and BD are defined)

(II) $(A\otimes B)^T=(A^T\otimes B^T)$ (where A^T is the transpose of A)

(III) $(A\otimes B)^{-1}=(A^{-1}\otimes B^{-1})$ (where A and B are non-singular matrices)

(IV) $\frac{d}{dt}(A\otimes B)=(\frac{dA}{dt}\otimes B)+(A\otimes\frac{dB}{dt})$

(V) $\|A\otimes B\|=\|A\|\|B\|$

For more information on Kronecker product of matrices, we refer to a paper of Murty and Fausett [14]. This paper is organized as follows: In section 2, we present certain important results on Kronecker product linear systems and the general solutions of (1.3) in terms of two fundamental matrix solutions of (1.1) and (1.2). We also obtain Φ-bounded solution of (1.1) and Ψ-bounded solution of (1.2) by using kronecker product of matrices in a systematic way. Here Φ and Ψ are ($m\times m$) and ($n\times n$) continuous matrices on R. The introduction of $(\Phi\otimes\Psi)$ function allows us to study stability and asymptotic stability of the system (1.3). In section 2, we present basic results that are needed for later discussion. Our main results are described in section 3. Solution of Kronecker product initial value problems was due to Murty, Balaram and Viswanadh [7]. Stability properties of a system of differential equations were studied by Murty, Viswanadh, *et. al.* [8].In 2019, Viswanadh and Murty presented a criterion for the existence and uniqueness of a solution to a three-point boundary value problem [6]. The existence of Ψ-bounded solutions of linear system of differential equations is presented in [1,2]. Kasi Viswanadh, *et. al.* [13] extended the idea of Ψ-bounded solutions of linear systems to time scale dynamical systems. These results are further extended to fuzzy systems by Kasi Viswanadh V. Kanuri in [12]. Further results on $(\Phi\otimes\Psi)$ are taken from [9]. Further significant results in this direction are also taken from [5,6,8]. For results on first order fuzzy difference systems, we refer to Yan Wu, Divya L. Nethi et.al [15]. The results presented in this paper can be extended to fuzzy Kronecker product linear systems and also on time scale dynamical systems. Working in this direction is in progress.

2. Preliminaries

In this section, we outline some of the basic results on Kronecker product linear systems and present Ψ-bounded solutions for a system of differential equations introduced by Aurel Diamandescu [1].

Theorem 2.1 $(\Phi(t)\otimes\Psi(t))$ is a fundamental matrix of

$$(x\otimes y)'=[(A(t)\otimes I_m)+(I_n\otimes B(t))](x\otimes y) \tag{2.1}$$

if and only if $\Phi(t)$ and $\Psi(t)$ are fundamental matrices of (1.1) and (1.2), respectively.
Proof: First suppose that $\Phi(t)$ and $\Psi(t)$ are fundamental solutions of (1.1) and (1.2). Then,

$$\Phi'(t)=A(t)\Phi(t) \text{ and } \Psi'(t)=B(t)\Psi(t).$$

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Consider,

$$\begin{aligned}(\Phi \otimes \Psi)' &= (\Phi' \otimes \Psi) + (\Phi \otimes \Psi') \\ &= (A\Phi \otimes \Psi) + (\Phi \otimes B\Psi) \\ &= [(A \otimes I_m) + (I_n \otimes B)](\Phi \otimes \Psi)\end{aligned}$$

Hence, $(\Phi \otimes \Psi)$ is a fundamental matrix of (2.1).
Conversely, suppose $(\Phi \otimes \Psi)$ is a fundamental matrix of (2.1). Then,

$$\begin{aligned}(\Phi(t) \otimes \Psi(t))' &= [(A(t) \otimes I_m) + (I_n \otimes B(t))](\Phi \otimes \Psi)(t) \\ &= [A(t)\Phi(t) \otimes \Psi(t)] + [(\Phi(t) \otimes B(t)\Psi(t)]\end{aligned}$$

from which,

$$[\Phi'(t) - A(t)\Phi(t)] \otimes \Psi(t) = \Phi(t) \otimes [B(t)\Psi(t) - \Psi'(t)].$$

Multiplying both sides with $\Phi^{-1}(t) \otimes \Psi^{-1}(t)$, we get

$$\Phi^{-1}(t)[\Phi'(t) - A(t)\Phi(t)] \otimes I_m = I_n \otimes \Psi^{-1}(t)[B(t)\Psi(t) - \Psi'(t)] \tag{2.3}$$

Equation (2.3) holds if and only if $\Phi^{-1}(t)[\Phi'(t) - A(t)\Phi(t)]$ and $\Psi^{-1}(t)[B(t)\Psi(t) - \Psi'(t)]$ are either a null matrix or unit matrix. If each equals a null matrix, then

$$\Psi^{-1}(t)[B(t)\Psi(t) - \Psi'(t)] = 0$$

implies $B\Psi(t) - \Psi'(t) = 0$. Hence, Ψ is a fundamental matrix of $y' = B(t)y$. Similarly, it can be shown that Φ is a fundamental matrix of $x' = A(t)x$. Suppose each expression is equal to a unit matrix, then

$$\Phi^{-1}(t)[\Phi'(t) - A(t)\Phi(t)] = I_n$$

$$\Phi'(t) = (A(t) + I_n)\Phi(t) \tag{2.4}$$

which clearly shows that Φ is a fundamental matrix of (2.4), a contradiction. Similarly, we can show that $\Psi^{-1}(t)[B(t)\Psi(t) - \Psi'(t)]$ cannot be a unit matrix.

Theorem 2.2 Let the matrix $A(t)$ be bounded. Then the system $x' = A(t)x$ has at least one Ψ-bounded solution in $\mathbb{R}^+$ for every Ψ-summable sequence of $\mathbb{R}^+$ if and only if there exists a positive constant K such that

$$\left\| \Psi(t)\Phi(t)P_1\Phi^{-1}(s)\Psi^{-1}(s) \right\| \le K \text{ for } 0 \le s \le t \tag{2.5}$$

and

$$\left\| \Psi(t)\Phi(t)P_2\Phi^{-1}(s)\Psi^{-1}(s) \right\| \le K \text{ for } 0 \le t \le s \tag{2.6}$$

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Definition 2.1 A function $\varphi : \mathbb{R}^+ \to \mathbb{R}^d$ is said to be Ψ-bounded on $\mathbb{R}^+$ if $\Psi(t)\varphi(t)$ is bounded on $\mathbb{R}^+$.

Let $A(t)$ be a continuous $(n \times n)$ matrix and $x' = A(t)x$, and let Φ be a fundamental matrix of $x' = A(t)x$ satisfying $\Phi(0) = I_n$. Let X_1 denote the subspace of $\mathbb{R}^n$ consists of all vectors whose values are Ψ-bounded solutions of $x' = A(t)x$, and let X_2 be an arbitrary fixed subspace of $\mathbb{R}^n$ supplementary to X_1. Let P_1 be the projection of $\mathbb{R}^n$ onto X_1 and $P_2 = I - P_1$ the projection on X_2. Then, $\lim_{t\to\infty} \|\Psi(t)\Phi(t)P_1\| = 0$ and $\lim_{t\to\infty} \|\Psi(t)\Phi(t)(I - P_1)\| = 0$. The Ψ-bounded solutions of linear system of differential equations ensures stability and asymptotic stability of the linear systems. If every solution of the linear system goes to zero as $t \to \infty$, then the linear system is said to be asymptotically stable.

3. Main results

In this section, we shall be concerned with the existence of $(\Phi \otimes \Psi)$ bounded solutions of the Kronecker product linear system (2.1 Φ is an $(n \times n)$ non-singular matrix and Ψ is an $(m \times m)$ non-singular matrix, and $(\Phi \otimes \Psi)^{-1} = \Phi^{-1} \otimes \Psi^{-1}$.Here after we assume that $Y(t)$ is a fundamental matrix of (1.1) and $Z(t)$ is a fundamental matrix of (1.2). We present a criterion for the existence of the $(\Phi \otimes \Psi)$ bounded solutions of the Kronecker product linear homogeneous equation (2.1).

Theorem 3.1 Let A and B be $(n \times n)$ and $(m \times m)$ continuous real matrices such that

$$\|(\phi(t) \otimes \psi(t))(Y(t) \otimes Z(t))(P_1 \otimes P_2)[Y^{-1} \otimes Z^{-1}][\phi^{-1}(s) \otimes \psi^{-1}(s)]\| \le K_1$$

, for $t \ge s \ge 0$

and let $\Phi(t)$, $\Psi(t)$ be such that

$$\|(\phi(t) \otimes \psi(t))(Y(t) \otimes Z(t))(I_n \otimes I_m - P_1 \otimes P_2)[Y^{-1} \otimes Z^{-1}][\phi^{-1}(s) \otimes \psi^{-1}(s)]\| \le K_2$$

, for all $t \ge s \ge 0$.

Then, the Kronecker product system (2.1) admits at least one $(\Phi \otimes \Psi)$ bounded solution on $\mathbb{R}^+$ for every continuous $(\Phi \otimes \Psi)$ bounded matrix function on $\mathbb{R}^+$ if and only if there exist positive constants K_1 and K_2 such that, for $0 \le s \le t$,

$$\left\|(\Phi(t) \otimes \Psi(t))(Y(t) \otimes Z(t))(P_1 \otimes P_2)(Y^{-1}(s) \otimes Z^{-1}(s))(\Phi^{-1}(s) \otimes \Psi^{-1}(s))\right\| \le (K_1 K_2) e^{-\alpha(t-s)} \quad (3.1)$$

for $0 \le t \le s$,

and

$$\left\|(\Phi(t) \otimes \Psi(t))(Y(t) \otimes Z(t))(P_1 \otimes P_2)(Y^{-1}(s) \otimes Z^{-1}(s))(\Phi^{-1}(s) \otimes \Psi^{-1}(s))\right\| \le (K_1 K_2) e^{-\alpha(s-t)} \quad (3.2)$$

for $0 \le s \le t$.

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Before we prove the theorem, we first observe that, from (3.1) and (3.2), as $t \to \infty$, the right-hand side of (3.1) and (3.2) approaches zero. Hence, every $(\Phi \otimes \Psi)$ bounded solution of (3.1) implies that the linear system is stable and infact asymptotically stable. We first prove the following lemma before proving our main theorem.

Lemma Let $A(t)$ and $B(t)$ be continuous real valued matrix functions such that the Kronecker product linear homogeneous system (2.1) has at least one $(\Phi \otimes \Psi)$ bounded solution on $\mathbb{R}^+$ for every $(\Phi \otimes \Psi)$ summable sequence f_n on $\mathbb{R}^+$ if and only if there exist positive constants K_1 and K_2 such that, for $0 \le s \le t$,

$$\|(\phi(t) \otimes \psi(t))(Y(t) \otimes Z(t))(P_1 \otimes P_2)[Y^{-1} \otimes Z^{-1}][\phi^{-1}(s) \otimes \psi^{-1}(s)]\| \le K_1$$

and for $0 \le t \le s$,

$$\|(\phi(t) \otimes \psi(t))(Y(t) \otimes Z(t))(P_1 \otimes P_2)[Y^{-1} \otimes Z^{-1}][\phi^{-1}(s) \otimes \psi^{-1}(s)]\| \le K_2$$

Proof: First, suppose that the Kronecker product system (2.1) has at least one $(\Phi \otimes \Psi)$ bounded solution on $\mathbb{R}^{m+n}$ for every continuous and $(\Phi \otimes \Psi)$ bounded sequence of functions $f_n(t)$. Moreover, let ***B*** be the Banach space of all $(\Phi \otimes \Psi)$ bounded and continuous functions $(x \otimes y): \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$, with the norm

$$\|(x \otimes y)\| = \sup_{t \in \mathbb{R}} \|(\Phi(t) \otimes \Psi(t))(x \otimes y)\|.$$

Let D denote the set of all $(\Phi \otimes \Psi)$ bounded and continuously differentiable functions

$(x \otimes y): \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$, such that $(x(0) \otimes y(0)) \in x_- \otimes x_+$

and $x' - Ax \in \boldsymbol{B}$ and $y' - By \in \boldsymbol{B}$. Evidently, D is a vector space. We define

$$\|x \otimes y\|_D = \|x \otimes y\|_B + \|(x * y)' - [(A(t) \otimes I_m)] \otimes [I_n \otimes B(t)]\|_B$$

Then, it can be easily proved that $\{D, \|.\|_D\}$ is a Banach space and

$$\lim_{n \to \infty} [\Phi(t) \otimes \Psi(t)][x_n(t) \otimes y_n(t)] = [\Phi(t) \otimes \Psi(t)][x(t) \otimes y(t)],$$

and hence,

$$\sup_{t \in \mathbb{R}} \|(\Phi(t) \otimes \Psi(t))(x * y)(t)\| \le K_1 \sup_{t \in \mathbb{R}} \|(\Phi(t) \otimes \Psi(t)) f(t)\|.$$

To this end, we define a mapping

$$T: D \to \boldsymbol{B} \text{ by } T(x \otimes y) = (x * y)' - (A \otimes I_m) \otimes (I_n \otimes B) \ ???$$

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Then, T is linear and bounded with $\|T\| \le 1$. It can be shown that T is onto. Hence, T^{-1} exists and is also bounded. The rest of the proof of the lemma follows.

Proof of Theorem 3.1: Suppose that the Kronecker product linear system (2.1) admits a $(\Phi \otimes \Psi)$ bounded solution. Then it is claimed that (3.1) and (3.2) hold. By hypothesis, there exists positive constants K_1 and K_2 such that

$$\int_0^t \left\| (\Phi(t) \otimes \Psi(t))(Y(t) \otimes Z(t))(P_1 \otimes P_2)(Y^{-1}(s) \otimes Z^{-1}(s))(\Phi^{-1}(s) \otimes \Psi^{-1}(s)) \right\| ds$$
$$+\int_t^\infty \left\| (\Phi(t) \otimes \Psi(t))(Y(t) \otimes Z(t))(P_1 \otimes P_2)(Y^{-1}(s) \otimes Z^{-1}(s))(\Phi^{-1}(s) \otimes \Psi^{-1}(s)) \right\| ds \le K_1 K_2$$

for $t \to 0$. Since Y and Z are fundamental matrix solutions of $x' = A(t)x$ and $y' = B(t)y$, it follows that

$$Y(t) = Y(s) - Y(0) + \int_s^t A(s)Y(s)ds\,,\ t,s \ge 0$$

and

$$Z(t) = Z(s) - Z(0) + \int_s^t B(s)Z(s)ds\,,\ t,s \ge 0\,.$$

Therefore,

$$(\Phi(t) \otimes \Psi(t))(Y^{-1}(s) \otimes Z^{-1}(s)) = (\Phi(t) \otimes \Psi(t))(\Phi^{-1}(s) \otimes \Psi^{-1}(s)) +$$
$$\int_s^t (\Phi(t) \otimes \Psi(t))[(A(s) \otimes I_m)(I_n \otimes B(s))][Y^{-1}(s) \otimes Z^{-1}(s)][\Phi^{-1}(s) \otimes \Psi^{-1}(s)]ds$$

Therefore, for all $t \ge s \ge 0$

$$\left\| (\Phi(t) \otimes \Psi(t))(Y(t) \otimes Z(t))(P_1 \otimes P_2)(Y^{-1}(s) \otimes Z^{-1}(s))(\Phi^{-1}(s) \otimes \Psi^{-1}(s)) \right\| \le K_1 K_2 e^{???(t-s)} \quad (3.3)$$

Now (2.5) and (3.3) imply (3.1). Similarly, (2.6) and (3.3) imply (3.2).
Conversely, suppose that (3.1) and (3.2) hold. Then, it is claimed that Y and Z are fundamental matrices to (1.1) and (1.2), respectively. From (3.1), we have

$$\begin{aligned}&\int_0^t \left\| (\Phi(t) \otimes \Psi(t))(Y(t) \otimes Z(t))(P_1 \otimes P_2)(Y^{-1}(s) \otimes Z^{-1}(s))(\Phi^{-1}(s) \otimes \Psi^{-1}(s)) \right\| ds \\ &+\int_t^s \left\| (\Phi(t) \otimes \Psi(t))(Y(t) \otimes Z(t))(P_1 \otimes P_2)(Y^{-1}(s) \otimes Z^{-1}(s))(\Phi^{-1}(s) \otimes \Psi^{-1}(s)) \right\| ds \le K_1 K_2\end{aligned} \quad (3.4)$$

since $e^{-\alpha(t-s)}$ is decreasing for all $t,s \ge 0$, $t - s > 0$, and $\alpha > 0$. Let $(u(t) \otimes v(t))$ be equal to the L.H.S. of (3.4). Then $(u(t) \otimes v(t))$ is a solution of (1.3). Hence, it follows that $(Y(t) \otimes Z(t))$ is a fundamental matrix solution of (1.3). By Theorem 2.1, it follows that Y is a fundamental matrix of (1.1) and Z is a fundamental matrix of (1.2). On the next theorem, we prove the asymptotic stability property of the solution of (1.3).

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Theorem 3.2 Suppose $(Y(t) \otimes Z(t))$ be a fundamental matrix solution of (1.3) satisfying (3.1) and (3.2). Then,

(i) $\lim_{t \to \infty} \|(\Phi(t) \otimes \Psi(t))(Y(t) \otimes Z(t))\| = 0$

(ii) The continuous $\Phi \otimes \Psi$ bounded functions $(f_1 \otimes f_2) : \mathbb{R}^+ \to \mathbb{R}^{mn}$ is such that $\lim_{t \to \infty} \|(\Phi(t) \otimes \Psi(t))(f_1 \otimes f_2)\| = 0$.

Proof: As a result of Theorem 3.1, we have for every continuous and $(\Phi \otimes \Psi)$ bounded functions $(f_1 \otimes f_2) : \mathbb{R}^+ \to \mathbb{R}^{mn}$, the equations (1.1) and (1.2) yield $(\Phi \otimes \Psi)$ bounded solution and hence the result follows.

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